MATH20132 Calculus of Several Variables. 2019-20

Problems 5: C¹-functions and more

C^1-***scalar***-valued functions

1. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(\mathbf{x}) = x \sin(xyz) + \exp(yz)$ where $\mathbf{x} = (x, y, z)^T$. Prove that f is a Fréchet differentiable function by showing that f is C^1 on $\mathbb{R}^3$.

2. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(\mathbf{x}) = \sin(xy^2z^3)$ where $\mathbf{x} = (x, y, z)^T$.

i. Prove that f is Fréchet differentiable at $\mathbf{a} = (\pi, 1, -1)^T$.

ii. Find the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ where $\mathbf{v} = (2/3, 1/3, -2/3)^T$.

The following was Questions 1& 2 on Sheet 4 but now, with C^1-functions, we can give a quicker solution.

3. a. By using partial differentiation find the gradient vectors of

i. $f : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto x(x+y)$ and

ii. $g : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto y(x-y)$

and show they are everywhere Fréchet differentiable. Find the directional derivatives of f and g at $\mathbf{a} = (1, 2)^T$ in the direction $\mathbf{v} = (2, -1)^T / \sqrt{5}$, justifying your method.

b. Using partial differentiation find the gradient vector of $h : \mathbb{R}^3 \to \mathbb{R}$ by $\mathbf{x} \longmapsto xy + yz + xz$ where $\mathbf{x} = (x, y, z)^T$, and show it is everywhere Fréchet differentiable. Find the directional derivative of f at $\mathbf{a} = (1, 2, 3)^T$ in the direction $\mathbf{v} = (3, 2, 1)^T / \sqrt{14}$, justifying your method.

4. (Tricky) *Recall:*

$$f \text{ is } C^1 \text{ at } \mathbf{a} \implies f \text{ is Fréchet differentiable at } \mathbf{a} \implies f \text{ continuous at } \mathbf{a}.$$

The contrapositive of this is

$$f \text{ not conts at } \mathbf{a} \implies f \text{ not F-differentiable at } \mathbf{a} \implies f \text{ is not } C^1. \quad (1)$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{xy}{x^2 + y^2} \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}; \qquad \text{with } f(\mathbf{0}) = 0.$$

This was shown in Question 11ii on Sheet 1 to **not** be continuous at $\mathbf{0}$. So, as not to contradict (1), prove that f is **not** C^1 at $\mathbf{0}$, i.e. that the partial derivatives are not continuous at $\mathbf{0}$.

*C^1-**vector**-valued functions*

5. Find the Jacobian matrices of the following functions, show that the functions are everywhere Fréchet differentiable and then find the directional derivatives at the given point $\mathbf{a}$ in the direction $\mathbf{v}$. In this way check your answers to Questions 5 & 7 on Sheet 3.

i. $\mathbf{f}\left((x, y, z)^T\right) = (xy, yz)^T$, $\mathbf{a} = (1, 3, -2)^T$ and $\mathbf{v} = (-1, 1, -2)^T / \sqrt{6}$,

ii. $\mathbf{f}\left((x, y)^T\right) = (xy^2, x^2y)^T$, $\mathbf{a} = (2, 1)$ and $\mathbf{v} = (1, -1)^T / \sqrt{2}$.

Chain Rule

6. Let

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2y \\ xy^2 \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} u + v \\ u - v \end{pmatrix},$$

for $\mathbf{x} = (x, y)^T$ and $\mathbf{u} = (u, v)^T$.

i. Calculate $\mathbf{f}\big(\mathbf{g}(\mathbf{u})\big)$ and thus find the Jacobian matrix $J(\mathbf{f} \circ \mathbf{g})(\mathbf{a})$ where $\mathbf{a} = (1, -2)^T$.

ii. Alternatively find $J\mathbf{f}(\mathbf{b})$, with $\mathbf{b} = \mathbf{g}(\mathbf{a})$, and $J\mathbf{g}(\mathbf{a})$ and use the Chain Rule to calculate $J(\mathbf{f} \circ \mathbf{g})(\mathbf{a})$

7. Use the Chain Rule to find the Fréchet derivative of $\mathbf{f} \circ \mathbf{g}$ at the given point $\mathbf{a}$ for each of the following.

i. i. With $\mathbf{x} = (x, y)^T$, $\mathbf{u} = (u, v)^T \in \mathbb{R}^2$,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2y \\ x - y \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} 3uv \\ u^2 - v \end{pmatrix},$$

at $\mathbf{a} = (2, 1)^T$.

ii. ii. With $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$, $\mathbf{u} = (u, v)^T \in \mathbb{R}^2$,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy \\ yz \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} uv^2 - v \\ u^2 \\ 1/uv \end{pmatrix},$$

at $\mathbf{a} = (2, 1)^T$.

8. Consider the Chain Rule in the case

$$\mathbb{R}^p \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{f} \mathbb{R},$$

so f is scalar-valued. Assume $\mathbf{g}$ is Fréchet differentiable at $\mathbf{a} \in \mathbb{R}^p$ and f is Fréchet differentiable at $\mathbf{b} = \mathbf{g}(\mathbf{a}) \in \mathbb{R}^m$. The Chain Rule says that $f \circ \mathbf{g}$ is Fréchet differentiable at $\mathbf{a}$ and $J(f \circ \mathbf{g})(\mathbf{a}) = Jf(\mathbf{b})\, J\mathbf{g}(\mathbf{a})$.

Think of the coordinates in $\mathbb{R}^p$ as x^i for $1 \leq i \leq p$, while in $\mathbb{R}^n$ they will be y^j for $1 \leq j \leq n$. Show that the Chain Rule can be written as

$$\frac{\partial f \circ \mathbf{g}}{\partial x^i}(\mathbf{a}) = \sum_{k=1}^{n} \frac{\partial f}{\partial y^k}(\mathbf{b}) \frac{\partial g^k}{\partial x^i}(\mathbf{a}),$$

for $1 \leq i \leq p$.

Extremal values of $d_{\mathbf{v}} f(\mathbf{a})$.

Here we find $\max_{\mathbf{v}:|\mathbf{v}|=1} d_{\mathbf{v}} f(\mathbf{a})$ *and* $\min_{\mathbf{v}:|\mathbf{v}|=1} d_{\mathbf{v}} f(\mathbf{a})$, *that is the directions of maximum and mimimum rate of change of f as we move away from* $\mathbf{a}$.

9. Suppose that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable on U and $\mathbf{a} \in U$. Prove that the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ has a

i. maximum value of $|\nabla f(\mathbf{a})|$ when $\mathbf{v}$ is in the direction of $\nabla f(\mathbf{a})$ and

ii. a minimum value of $-|\nabla f(\mathbf{a})|$ when $\mathbf{v}$ is in the direction of $-\nabla f(\mathbf{a})$.

Hint for any vectors we have $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

10. Suppose the temperature at a point $(x, y, z)^T$ in a metal cube is given by

$$T = 80 - 60xe^{-\frac{1}{20}\left(x^2+y^2+z^2\right)},$$

where the centre of the cube is taken to be $(0, 0, 0)^T$. In which direction from the origin is the rate of change of temperature greatest? The least?

Additional Questions 5

11 Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $\mathbf{x} \mapsto xy^2z$.

i. Show that f is a C^1-function on $\mathbb{R}^3$.

ii. Calculate $\nabla f(\mathbf{a}) \bullet \mathbf{v}$ with $\mathbf{a} = (1, 3, -2)^T$ and $\mathbf{v} = (-1, 1, -2)^T / \sqrt{6}$. Explain any similarity with Question 4 Sheet 3.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{\sin(x^2y^2)}{x^2 + y^2} \quad \text{if } \mathbf{x} = (x, y)^T \neq \mathbf{0}; \qquad f(\mathbf{0}) = 0.$$

i. Find the partial derivatives of f at **all** points $\mathbf{x} \in \mathbb{R}^2$.

   Hint For $\mathbf{x} = \mathbf{0}$ you will have to return to the definition of partial derivative.

ii. Prove that f is a C^1-function on $\mathbb{R}^2$ with Fréchet derivative $df_{\mathbf{0}} = \mathbf{0} : \mathbb{R}^2 \to \mathbb{R}$ at the origin.

   Hint You may make use of $|\sin\theta| \leq |\theta|$ for all θ.

13. *Further practice on the Chain Rule* Use the chain rule to find the derivative of $\mathbf{f} \circ \mathbf{g}$ at the point $\mathbf{c}$ for each of the following. Give your answers in the form $d(\mathbf{f} \circ \mathbf{g})_{\mathbf{c}}(\mathbf{t})$.

i. $\mathbf{f}\big((x, y)^T\big) = (x^2y,\ x-y)^T$, $\mathbf{g}\big((u, v)^T\big) = (3uv,\ u^2-4v)^T$, $\mathbf{c} = (1, -2)^T$,

ii. $\mathbf{f}\big((x, y, z)^T\big) = (4xy,\ 3xz)^T$, $\mathbf{g}\big((u, v)^T\big) = (uv^2 - 4v,\ u^2,\ 4/uv)^T$, $\mathbf{c} = (-2, 3)^T$

iii. $\mathbf{f}\big((x, y)^T\big) = (3x+4y,\ 2x^2y,\ x-y)^T$, $\mathbf{g}\big((u, v, w)^T\big) = (4u-3v+w,\ uv^2)^T$, $\mathbf{c} = (1, -2, 3)^T$.

14. Revisit Question 17iii on Sheet 3. Define the functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ by $(x, y)^T \mapsto (x + y, x - y, xy)^T$ and $h : \mathbb{R}^3 \to \mathbb{R}$ by $(x, y, z)^T \mapsto xy^2z$. Calculate, using the Chain Rule, the directional derivative of $h \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$ at $\mathbf{a} = (2, -1)^T$ in the direction $\mathbf{v} = (1, -2)^T / \sqrt{5}$.

15. Assume $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is Fréchet differentiable at $\mathbf{q} = (2, 3)^T$ with

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ 0 & 4 \end{pmatrix}.$$

Assume also that $\mathbf{F}(\mathbf{q}) = \begin{pmatrix} 2 & -1 & 4 \end{pmatrix}^T$.

Define $f : \mathbb{R}^2 \to \mathbb{R} : f(\mathbf{x}) = |\mathbf{F}(x)|$. Prove that f is Fréchet differentiable at $\mathbf{q}$ and find $df_{\mathbf{q}}(\mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^2$.

16. A heat-seeking insect always moves in the direction of the greatest increase in temperature. Describe the path of a heat-seeking insect placed at $(1, 1)^T$ on a metal plate heated so that the temperature at $\mathbf{x} = (x, y)^T$ is given by

$$T(\mathbf{x}) = 100 - 40xye^{-r(\mathbf{x})},$$

where $r(\mathbf{x}) = (x^2 + y^2)/10$.

What if the insect starts at $(3, 2)^T$? Or the origin $\mathbf{0}$?